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Optimization of Frequencies Spectrum in Vibrations of Flexible Structures

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Optimality criteria are derived and used to maximize a set of frequencies for a structure of given weight. The criteria include both membrane and bending effects and nonlinear relations between the stiffness and mass matrices. An error norm is proposed and used to determine the values of design variables and Lagrange multipliers at optimum. Some details of the accompanying iterative procedure are discussed and the numerical examples of simple test structures are presented.

Introduction

STRUCTURES with some frequency requirements can be optimized using two computationally different strategies. Either the weight of the structure is minimized subject to given frequency constraints, or the frequencies are maximized for structures of given weight.

The first strategy requires scaling procedures in order to adjust the frequencies after each iteration to the level specified by the constraints.[1-4] Since the frequencies are, in general, complicated functions of the design variables, these procedures may pose a serious challenge, particularly, if the element stiffness matrices are nonlinear in terms of the design variables or if multiple eigenvalues are involved.

The second strategy,[5-8] in which the frequencies are treated as objective functions, though somewhat more analytically involved (inclusion of more than one mode of frequency into consideration automatically creates a multicriteria optimization problem), seems to be free of the difficulties mentioned above. In particular, scaling of the weight constraint after each iteration is either trivial or not necessary at all.[7,8] Also handling multiple eigenvalues is relatively straightforward, as it was shown in Refs. 5, 7, and 8. The optimal (maximum) fundamental frequencies found there were either inherently multimodal or became multimodal in the course of optimization process.

Theoretically, these two strategies are equivalent because the structure treated either way must satisfy identical optimality conditions at optimum.[6,7]

The second strategy is used here to maximize a cluster of the first N frequencies for structures of given weight and with specified ratios between these frequencies. The method proposed is an extension of the approach presented in Refs. 7 and 8 where only the fundamental frequency was maximized.

We used the optimality criteria method which is considered independent of the number of design variables and is particularly well suited for optimal sizing of structures.[9]

The optimality criteria derived here are used to formulate an error norm which represents the distance between any trial design and the optimal design. The iterative procedure is then developed that modifies design variables in order to reduce this error norm. If the error norm is sufficiently small, the design satisfies all of the optimality conditions and is assumed optimal.

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Optimization Problem

Consider an elastic structure of a given weight W_0 (volume V_0). The set of N frequencies of free vibrations $\omega_1, \ldots, \omega_N$ is to be monitored and optimized to improve the dynamic characteristics of the design without adding to its weight. It should be done by modifying the stiffness and mass of all of the structural members until the structure meets specified optimality conditions.

In general, we want to maximize the whole cluster of frequencies in such a way that the ratio of each higher frequency to the fundamental one satisfies the condition

$$\omega_i/\omega_1 \geq 1 + a_i \tag{1}$$

where a_i is a given set of coefficients indicating the required distribution of the frequencies in the cluster. The coefficients a_i may be multiple; however, they all must be positive and in ascending order (i.e., $a_i \leq a_{i+1}$). Formally, the objectives of the optimization are stated as

maximize

$$\omega_1 \tag{2a}$$

subject to

$$(1 + a_i)\omega_1 - \omega_i \leq 0, \qquad i = 2, \ldots, N \tag{2b}$$

and

$$\sum_j V_j = V_0 \tag{2c}$$

also

$$V_{\min} \leq V_j \leq V_{\max} \tag{2d}$$

where design variable V_j is the volume of jth element and the summation is over all of the structural elements used in the FEM analysis. Since, formally, the objective function ω_1 is used in the constraints [Eq. (2b)], this problem can also be considered as a multicriteria optimization problem.[5,6]

Equation (2c) represents a constant-volume constraint and Eq. (2d) are side constraints. The side constraints can be enforced explicitly in the course of iterations and do not need to be included in the optimality criterion.

If all a_i are zero, this represents the case of multimodal optimization for maximum fundamental frequency discussed in Refs. 7 and 8. For $a_i = \ldots = a_{i+l}$ (where $i + l \leq N$), the ith frequencies may be multiple while the remaining frequencies, $\omega_1, \ldots, \omega_{i-1}$, may be of single modes.

We assume that all of the frequencies required and the corresponding eigenvalues x_i can be obtained solving the eigenvalue problem:

$$(K - \omega_i^2 M) x_i = 0 \qquad (3)$$

where K and M are the stiffness and mass matrices, respectively.

The eigensolutions satisfy the following orthogonality and normalization properties:

$$½(x_i^T Mx_l) = \delta_{il} \qquad (4a)$$

$$½(x_i^T Kx_l) = \omega_i^2 \delta_{il} \qquad (4b)$$

where δ_{il} is the Kronecker delta and the factor ½ is added for convenience.

Optimality Conditions

The Lagrange functional for the problem [Eq. (2)] is written in the form

$$L(x_i, V_j) = \omega_1^2 + \sum_{i=2}^{N} \gamma_i [\omega_i^2 - (1+a_i)^2 \omega_1^2] - \sum_{i=1}^{N} \eta_i \left(\frac{1}{2} x_i^T Mx_i - 1\right) + \beta\left(\sum_j V_j - V_0\right) \qquad (5)$$

where γ_i are Lagrange multipliers related to the inequalities in Eq. (2b) and η_i and β are the multipliers corresponding to the equality constraints given by Eq. (4a) and Eq. (2c), respectively. The index "i" represents the ith mode and the index "j" represents the jth element throughout the paper.

Note that in order to maximize the objective function [Eq. (2a)], the multipliers γ_i ($i = 2, \ldots, N$) cannot be negative. Taking the derivatives of L with respect to x_i and using Eq. (4), one obtains the equations identical in the form to Eq. (3) from which we can conclude that

$$\eta_i = \gamma_i \omega_i^2, \qquad i = 1, \ldots, N \qquad (6a)$$

where

$$\gamma_1 = 1 - \sum_{i \geq 2}^{N} (1+a_i)^2 \gamma_i \qquad (6b)$$

The multiplier γ_1 introduced here and related to the frequency ω_1 (the corresponding $a_1 = 0$) is not independent. It can be determined only after $\gamma_2, \ldots, \gamma_N$ are found. Using this multiplier, the Lagrange functional can be rewritten in the alternative form

$$L(x_i, V_j) = \sum_{i=1}^{N} \gamma_i \omega_i^2 - \sum_{i=1}^{N} \eta_i \left(\frac{1}{2} x_i^T Mx_i - 1\right) + \beta\left(\sum_j V_j - V_0\right) \qquad (7)$$

From the viewpoint of multicriteria optimization theory,[10] the functional Eq. (7) represents the problem of finding Pareto-optima for the linear combination of N functions ω_i with the corresponding weighting factors γ_i satisfying Eq. (6b). This dual interpretation of γ_i [in Eq. (5) they were treated strictly as Lagrange multipliers] can conveniently be used to raise separately any single frequency by manipulating the magnitudes of γ_i. For example, if one a priori assumes that $\gamma_k = 1$ for a particular kth mode, and $\gamma_i = 0$ for $i \neq k$, only this kth frequency will be maximized. Similarly, if $\gamma_2 = \ldots \gamma_n = 0$, from Eq. (6b) we have $\gamma_1 = 1$ which is the case of optimization for the fundamental frequency ω_1. However, this optimization is valid only for unimodal optimal designs. Such cases were considered in Refs. 4, 11, and 12. Note that Eq. (6a) can be substituted into Eqs. (5) or (7) only after the derivatives of L with respect to V_j are calculated. This is necessary in order to distinguish between ω_i in Eq. (5) which is considered a function of V_j and ω_i in Eq. (6a) treated as a parameter characterizing a design of a given set of V_j.

Taking the derivatives of L with respect to V_j and then substituting Eq. (6a), we have

$$\frac{\partial L}{\partial V_j} = \sum_i \gamma_i \left(\frac{1}{2} x_i^T \frac{\partial K}{\partial V_j} x_i - \frac{1}{2} \omega_i^2 x_i^T \frac{\partial M}{\partial V_j} x_i\right) + \beta = 0 \qquad (8)$$

For most of the elements (beams, plates for example), the stiffness matrix for the jth element K_j can be decomposed into the part representing the membrane (axial) K_j^a and the bending K_j^b stiffnesses such that

$$K_j = K_j^a + K_j^b \qquad (9)$$

This decomposition is necessary because the membrane and bending stiffnesses are related in different ways to the geometrical parameters of the elements that are used as design variables (height, width, cross-sectional area, etc.). For example, for a two-dimensional beam element, we have

$$K_j = c_j^a A_j + c_j^b I_j \qquad (10)$$

where c_j^m and c_j^b are matrices depending on the material properties and the length of the element only, and A_j and I_j are the cross-sectional area and moment of inertia, respectively. To use, for example, this cross-sectional area A_j as design variables, we have to define the relation $I(A)$ which usually is assumed in the form

$$I_j = b\,(A_j)^p \qquad (11)$$

where b and p are constant parameters. Typically, $p = 1$ (truss element), 2, or 3 if the width, height, or both are to be modified. Note that p can also be a real number if only a portion of the cross section (a flange for example) is optimized.

The mass matrix for the jth element can be written in the form

$$M_j = c_j^m A_j + m_j \qquad (12)$$

where c_j^m is a matrix depending on the material properties and the length of the element and m_j is a nonstructural portion of the mass. We use either structural elements for which $m_j = 0$ or nonstructural elements for which $M_j = m_j$. Now, using Eqs. (9–12), the derivatives in Eq. (8) are determined as

$$\frac{\partial K}{\partial V_j} = (1-p)\frac{c_j^a}{l_j} + \frac{K_j}{V_j} \qquad (13a)$$

$$\frac{\partial M}{\partial V_j} = \frac{M_j}{V_j} \qquad (13b)$$

for structural elements only, and where l_j denotes the length of the element.

Substituting Eq. (13) into Eq. (8) we have

$$\frac{\partial L}{\partial V_j} = \sum_{i=1}^{N} \gamma_i [(1-p)\, NE_{ij} + pSE_{ij} - KE_{ij}] + \beta\, V_j = 0 \qquad (14)$$

where

$$NE_{ij} = ½\ [x_{ij}^T (c_j^a A_j)\, x_{ij}] \qquad (15a)$$

$$SE_{ij} = ½\ (x_{ij}^T K_j x_{ij}) \qquad (15b)$$

$$KE_{ij} = ½\ \omega_i^2 (x_{ij}^T M_j x_{ij}) \qquad (15c)$$

Here NE_{ij}, SE_{ij}, and KE_{ij} are the membrane (axial) strain energy, the total strain energy, and the kinetic energy, respectively, stored in the jth element due to the ith mode of vibrations.

Summing up the energies in all elements, we obtain

$$\sum_j SE_{ij} = \sum_j KE_{ij} = \omega_i^2 \tag{16}$$

For each mode, we can determine the ratio of the membrane strain energy to the total strain energy ϕ_i defined by

$$\phi_i = \sum_j NE_{ij} \Big/ \sum_j SE_{ij} \tag{17}$$

and the ratio of kinetic energy of nonstructural elements to the total kinetic energy, α_i, given by

$$\alpha_i = \sum_j x_{ij}{}^T m_j x_{ij} \Big/ \sum_j x_{ij}{}^T M_j x_{ij} \tag{18}$$

Summing Eq. (4) for all elements and using Eqs. (15–18), the multiplier β can be determined as

$$\beta = -\sum_{i=1}^{N} \gamma_i \omega_i^2 [(1-\phi_i)(p-1)+\alpha_i]/V_0 \tag{19}$$

Note that $\omega_i = (1+a_i)\omega_1$ for all nonzero multipliers γ_i. Using this, Eq. (14) can be written in the form

$$\frac{\partial L}{\partial V_j} = \sum_{i=1}^{N} \gamma_i (1+a_i)^2 (EES_{ij} - MC_i) = 0 \tag{20}$$

where

$$EES_{ij} = [(1-p)NE_{ij} + pSE_{ij} - KE_{ij}]/AE_{ij} \tag{21a}$$

$$MC_i = (1-p)\phi_i + p + \alpha_i - 1 \tag{21b}$$

and where

$$AE_{ij} = \left(\sum_j SE_{ij}\right) V_j/V_0 \tag{22}$$

Here AE_{ij} is the "average" strain energy stored in the jth element due to the ith vibrations mode. EES_{ij} is the element energy state in the jth element due to the ith mode, and MC_i is the modal constant for the ith mode.

The fundamental difference between a single-mode optimization ($N=1$) and a multimodal optimization ($N>1$) should be noted. This difference will effect any perspective iterative scheme. If dealing with a single-mode optimization for which $\gamma_2 = \gamma_n = 0$, from Eq. (20) one obtains

$$EES_{1j} = MC_1 = \text{const} \tag{23}$$

proving that the combination of energies as defined in Eq. (21a) must be identical for all elements. (This approach was also called the energy ratio method.[13]) For multimodal optimization, the magnitude of EES_{ij} at optimum varies from

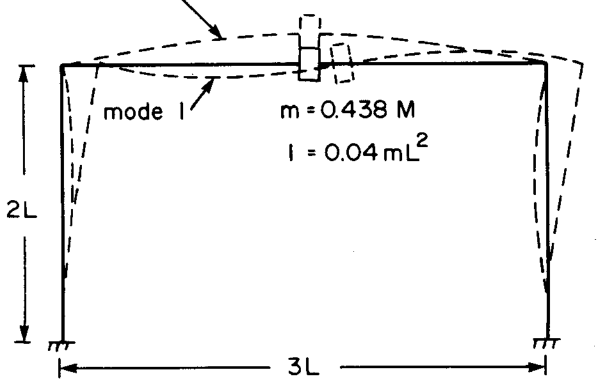

Fig. 1a Plane test frame.

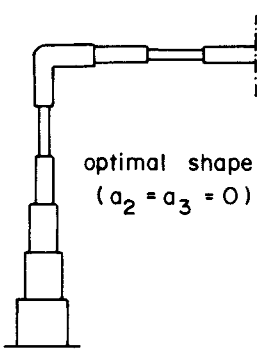

Fig. 1b Sizing at bimodal optimum.

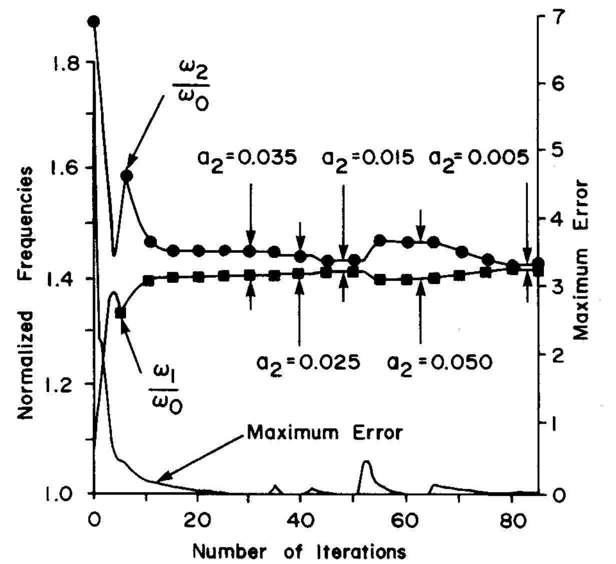

Fig. 1c Optimization histogram for various mode separations.

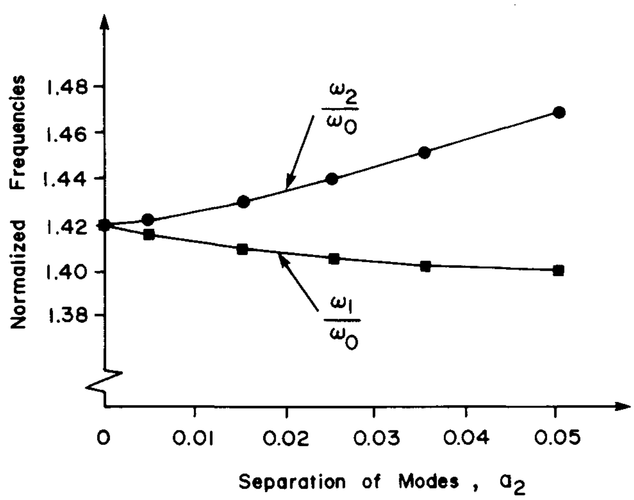

Fig. 1d Effect of mode separations on frequencies at optimum.

element to element and obviously is not equal to MC_i. Additionally, the set of γ_i is to be determined.

To obtain the complete set of Kuhn-Tucker necessary conditions for the problem, Eq. (2), we have to add the following switching conditions to Eq. (20), which represents the L_v gradient:

$$\gamma_i\,[(\omega_i^2/\omega_1^2)-(1+a_i)^2]=0,\quad i=2,\ldots,N \tag{24}$$

Additionally, the Lagrange multipliers must be positive and must satisfy Eq. (6b). Summarizing, the complete set of Kuhn-Tucker conditions reads

$$(K-\omega_i^2M)\,x_i=0,\qquad i=1,\ldots,N \tag{25a}$$

$$\sum_{i+1}^{N}\gamma_i(1+a_i)^2\,(EES_{ij}-MC_i)=0$$
$$j=1,\ldots,EN \text{ (all elements)} \tag{25b}$$

$$\gamma_i\,[\omega_i^2/\omega_1^2-(1+a_i)^2]=0,\qquad i=2,\ldots,N \tag{25c}$$

$$\sum_{i=1}^{N}\gamma_i\,(1+a_i)^2=1,\qquad \text{and}\qquad \gamma_i\geq 0 \tag{25d}$$

The first equation, Eq. (25a), represents the L_x gradient and is always satisfied automatically at the beginning of each iteration when solving the eigenvalue problem. Thus, this equation is used only to update ω_i and x_i in terms of the current magnitudes of design variables. The remaining conditions, Eqs. (25b-25d), altogether $EN+N$ equations, will be used to determine EN values of V_j and N values of γ_i at optimum.

It can be also shown that the Lagrange functional L is at least a positive semidefinite function of V_j since

$$\frac{\partial^2 L}{\partial V_j \partial V_k}=0,\qquad \text{for } j\neq k \tag{26a}$$

$$\frac{\partial^2 L}{\partial V_j^2}=\frac{p(p-1)}{2V_j^2}\sum_{i=1}^{N}\gamma_i\,(1+a_i)^2\,[x_{ij}^T\,K_j^b\,x_{ij}]\geq 0 \tag{26b}$$

The term in the brackets represents the bending strain energy stored in the jth element and cannot be negative. Thus, L is a positive definite function of V_j except for the case when only the membrane (axial) strain energy is involved as for example for trusses (for which also $p=1$).

It can be also shown that any constraints, Eq. (2b), or any equivalent constraint $g_i(V_j)$ in the form

$$g_i=(1+a_i)^2\,\omega_1^2-\omega_i^2\leq 0 \tag{27}$$

is also at least a positive semidefinite function of V_j. Therefore, the optimization problem discussed here can be classified as a convex problem and consequently any design satisfying the conditions [Eq. (25)] must be at global optimum. Because of convexity of the optimization problem, the solution to these equations should be unique.

Iterative Procedure

Having derived the optimality criterion, we can use it to verify any trial design (provided the modal analysis is completed) against optimum.

The following error norm Ω is proposed:

$$\Omega=\frac{1}{2V_0}\sum_j(\Psi_j)^2\,V_j+\frac{1}{2}\sum_{i=1}^{N}\gamma_i^2\,[(\omega_i^2/\omega_1^2)-(1+a_i)^2]^2 \tag{28a}$$

where

$$\Psi_j=\sum_{i=2}^{N}\gamma_i\,(1+a_i)^2\,(EES_{ij}-MC_i) \tag{28b}$$

Here Ψ_j are residuals of Eq. (25b) and the second term represents the error in Eq. (25c). Only γ_i meeting the requirement [Eq. (25d)] are admitted. The residuals Ψ_i satisfy the following relation [this can be directly proved using Eqs. (17), (21–22), and (28b)]:

$$\sum_j\Psi_j\,V_j=0 \tag{29}$$

Clearly the norm Ω is always positive except for the optimal point where all conditions [Eq. (25)] are satisfied and where it becomes zero.

If, from the modal analysis of a trial design, the frequencies ω_i and the eigenmodes x_i are found, one can select any set of γ_i satisfying Eq. (25d), substitute it into Eq. (28), and calculate $\Omega(\gamma_i)$. This is possible due to the dual interpretation of γ_i [see Eq. (7)]. This property can also be used in unimodal optimization to maximize a particular frequency ω_k. It is enough to assume $\gamma_i=0$, where $i\neq k$, for this purpose. To solve the general problem [Eq. (2)], we will select a set of γ_i that minimizes Ω (which is a quadratic function of γ_i). It can be done by setting to zero the derivatives of Ω with respect to γ_i, that is;

$$\frac{\partial\Omega}{\partial\gamma_i}=\frac{1}{V_0}\sum_j\Psi_j\cdot\frac{\partial\Psi_j}{\partial\gamma_i}V_j+\gamma_i\left[\left(\frac{\omega_i^2}{\omega_1^2}\right)-\left(1+a_i\right)^2\right]^2=0,$$
$$i=2,\ldots,N \tag{30}$$

Substituting Eq. (28b), we get $N-1$ linear equations with respect to $\gamma_2,\ldots,\gamma_n$. Calculating γ_i from Eq. (30) and substituting back into Eq. (28a) should give us the minimum value of the error norm Ω. However, the set of γ_i satisfying Eq. (30) may not meet the requirements [Eq. (25d)], especially at the beginning of iterations when a trial design is still quite distant from optimum.

These requirements can be enforced numerically modifying the γ_i ($i=2,\ldots,N$) calculated from Eq. (30) by setting all negative multipliers to zero and then scaling down the remaining γ_i to satisfy Eq. (25d). Such modified set of γ can then be used to obtain the residuals Ψ_j for each element and finally to calculate the magnitude of Ω.

Since Ψ_j satisfy Eq. (29), which states that the weighted average residual is zero, we can conclude that all nonoptimal elements can be separated into two categories having either positive or negative residuals. The elements can be either too stiff or not stiff enough, therefore it is easy to find a correspondence between the sign of the residual and the stiffness of each particular element. Numerical experimentations (there are only two options to check!) as well as some analysis[7] prove that elements with positive Ψ_j are too flexible and elements with negative Ψ_j are too stiff.

This information suggests the character of modifications to the design variables. For example, using the cross-sectional area A_j, too flexible elements will be stiffer and too stiff elements will be made more flexible if the area increments ∂A_j are assumed in the form

$$\partial A_j=c\,\Psi_j\,A_j \tag{31}$$

where c is a positive number. Further analysis[7] shows that if c is small enough, typically $c\approx 0.1$, and if the changes of γ_i between iterations are negligible, the residual increment $\partial\Psi_j$, to be expected in the next iteration, should approximately be equal to

$$\partial\Psi_j\cong -c\,\Psi_j \tag{32}$$

Since $c>0$, the signs of the increment $\partial\Psi_j$ and the residual Ψ_j are always opposite. This feature indicates that the procedure should perform somewhat similar to the performance of the gradient-based procedures. It is also important that the residuals Ψ_j will gradually be reduced to zero independently of the selection of the set of γ_i, which is one of the consequences

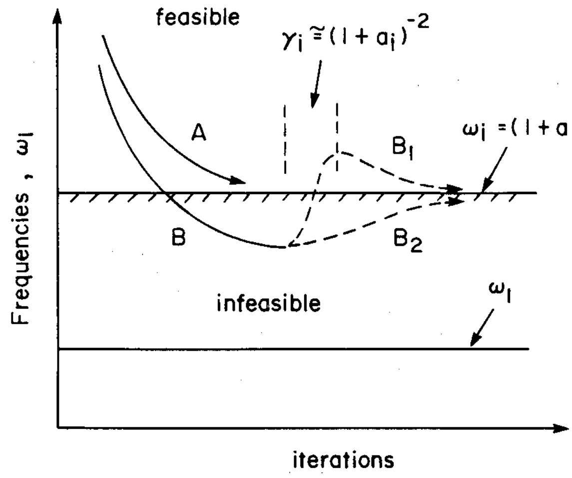

Fig. 2 Possible optimization paths.

of the dual formulation [Eq. (7)]. However, if Eq. (30) is used for this purpose, the resulting set of γ_i will finally satisfy the switching condition [Eq. (24)], since in the limit $\Psi_j \to 0$ and Eqs. (25c) and (30) will become equivalent.

Additionally, if the modifications to the design variables A_j are such that the volume increment ∂V_j is proportional to the residual Ψ_j, as it is the case when using Eq. (31), the resultant total volume increment, because of the property Eq. (29), is always zero, that is,

$$\partial V = \sum_j \partial V_j = c \sum_j \Psi_j V_j = 0 \tag{33}$$

This relation proves that between iterations no scaling of the design variables is required in order to meet the equality constraint since the volume of the design remains automatically constant. Numerically, this is a very convenient feature of the procedure.

Some Numerical Tests and Discussion

The procedure discussed here has been tested on simple beam, frame, and plate structures. For the eigenvalue analysis ANSYS, the FEM program was used. ANSYS, besides the complete eigensolution, provided also the total strain energy and the kinetic energy for elements due to each vibration mode [SE_{ij} and KE_{ij} in Eqs. (15b) and (15c)]. Only the membrane strain energy NE_{ij} had to be calculated using Eq. (15a) directly. The initial designs were always uniform with the fundamental frequency denoted by ω_o.

Figure 1 shows a frame fixed to the ground with a nonstructural mass in the middle. This frame, which was also used to test optimization procedures presented in Refs. 7 and 12, is convenient for a multimodal testing because the frequencies corresponding to the first two modes are much lower than the frequency of the third mode and because the relative magnitudes of these two frequencies (for the initial design) can be easily adjusted by using smaller or bigger nonstructural mass m. Therefore, at optimum, one should expect that the second mode will be included (bimodal optimum design) and the third mode will be excluded. This somehow predictable optimization route can be monitored in terms of convergence and behavior of Lagrange multipliers.

The frame shown in Fig. 1a is made of aluminum and weighs $M = 0.8$ kg. When uniform it has the following frequencies: $\omega_o = \omega_1 = 130.7$ Hz (antisymmetric), $\omega_2 = 1.88\ \omega_o = 246$ Hz (symmetric), and $\omega_3 = 7.47\ \omega_o = 971$ Hz (antisymmetric). The frequency spectrum is maximized assuming various separation ratios between the first and higher modes. Figure 1c shows the iterations history if the separation was first 3.5% ($a_2 = 0.035$, $a_3 = 0.07$), then was changed to 2.5% ($a_2 = 0.025$, $a_3 = 0.05$), and so on.

As expected, the third mode was too high to influence the optimum; therefore, its histogram is not shown in the plot. Clearly, if the fundamental frequency is the primary goal of optimization, its maximum value ($\omega_{max} = 1.42\ \omega_o = 186$ Hz) can be obtained assuming that all $a_i = 0$. Any requested separation $a_2 > 0$ will lower ω_1 as it is shown in Fig. 1d.

It should be emphasized that, according to the formulation used here, a trial design for which $\omega_i < \omega_1 (1 + a_i)$ is infeasible [see Eq. (27)]. Such a possibility exists only if $a_i > 0$. If all $a_i = 0$, due to the ascending order of the eigenvalue solutions, any trial design must satisfy the constraint Eq. (2b) and consequently the optimum will always be approached from the feasible direction with $\gamma_i \geq 0$. If, however, $a_i > 0$, there are, in general, two possibilities:

1) Optimum does not exist, which means that the assumed a_i is too high for this particular design and consequently all trial designs will be consistently infeasible.

2) Optimum exist and can be approached from feasible (line A in Fig. 2) or infeasible directions (line B).

In the latter case, there are several possible scenarios how the optimum may be reached. If for a trial design $\omega_i < \omega_1 (1 + a_i)$, this design can formally be converted to feasible if the constraint function g_i instead of being considered nonpositive, as in Eq. (27), is assumed to be nonnegative [e.g., $g_i(V_j) \geq 0$]. However, as a consequence of this redefinition, the corresponding γ_i must be nonpositive at optimum which means that the iterative procedure must be adjusted so as to accept and process negative Lagrange multipliers (line B_2).

Another option is to drive the design into the feasible domain first and then approach the optimum. This can be done by modifying γ_i for the next iteration to the value that should be higher than the one calculated from Eq. (30), but not bigger than $\gamma_i = (1 + a_i)^{-2}$. If one assumes $\gamma_i = (1 + a_i)^{-2}$ [the remaining multipliers must be zero according to Eq. (25d)], the procedure will attempt to raise the frequency ω_i only [see Eq. (7)]. The modified γ_i should be used until ω_i becomes bigger than $(1 + a_i)\omega_1$ (line B_1 in Fig. 2).

If the modification of γ_i are in the form

$$(\gamma_i)_{\text{modified}} = \gamma_i\ [\omega_1 (1 + a_i)/\omega_i]^h \tag{34}$$

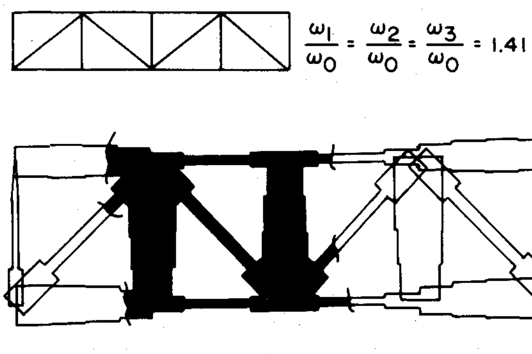

Fig. 3a Free-flying test frame, trimodal optimum.

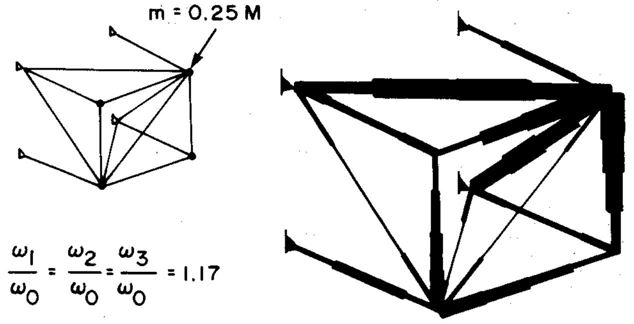

Fig. 3b Spatial test frame, trimodal optimum.

where h is a positive variable exponent, this generally speeds up the convergence. The values $h \cong 1$, if the current trial design is feasible, and $h \geq 10$, if the design is infeasible, are recommended. Note that the modification Eq. (34) acts toward closing the gap between ω_i and ω_1 if $\omega_i > (1 + a_i)\omega_1$ and makes this gap bigger if $\omega_i < (1 + a_i)\omega_1$.

Two other examples of optimal designs are shown in Fig. 3. These frames were optimized for maximum fundamental frequency (all $a_i = 0$). Both frames are characterized by a densely packed cluster of frequencies. For example, initially (uniform) the first five frequencies for the free-flying frame (Fig. 3a) were 22.5, 22.7, 25.6, 27.3, and 29.3 Hz; thus the fifth frequency ω_5 was only 30.4% higher than the fundamental one. At optimum the fundamental frequency has been raised to 31.7 Hz (41% increase). This frequency is trimodal. The three-dimensional frame shown in Fig. 3b is also trimodal at optimum, although its fundamental frequency has been increased only by 17.4%.

Conclusions

An alternative strategy in which the eigenvalues are used as objective functions for optimization of structures with frequency requirements is presented. From the engineering viewpoint, this strategy may be perhaps less intuitive and natural than the minimum weight optimization; however, it provides some numerical advantages discussed in the paper. The optimal set of design variables is determined satisfying the complete optimality conditions. The iterative procedure proposed does not require any constraints, approximations, or scaling of the design variables.

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